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Variational equations and symmetries in the Lagrangian formalism

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Abstract. Symmetries in the Lagrangian formalism of arbitrary order are analysed with the help of the so-called Anderson–Duchamp–Krupka equations. For the case of second-order equations and a scalar field we establish a polynomial structure in the second-order derivatives. This structure can be used to clarify the form of a general symmetry. As an illustration we analyse the case of Lagrangian equations with Poincaré invariance or with universal invariance.

1. Introduction

The study of classical field theory in the framework of the Lagrangian formalism is still a subject of active research. For first-order Lagrangian systems one usually prefers the use of the Poincaré–Cartan form or related geometrical objects (see for instance [1, 2]). For higher-order Lagrangian systems it is difficult to find a proper generalization of the Poincaré–Cartan form with the same properties as the first-order case. It is particulary difficult to find such a generalization having a nice behaviour with respect to the (Noetherian) symmetries. A way out is to use a related formalism based on the Euler–Lagrange operator and its intrinsic characterization by Helmholtz equations. In fact, it was noticed sometimes ago that, in the case of second-order differential equations describing a system with a finite number of degrees of freedom, one can give necessary and sufficient conditions such that the equations follow from a Lagrangian: they are the so-called Helmholtz equations (see [3] for a complete bibliography on this problem). Remarkably, this result can be extended to the general case of classical field theory and to equations of arbitrary order, leading to the so-called Anderson–Duchamp–Krupka (ADK) equations [4, 5], which seem to be less well known in the physics literature. The proper framework for this formalism is based on the jet-bundle structures.

The purpose of this paper is to prove that this formalism based on the ADK equations can be used to treat higher-order Lagrangian systems with groups of symmetries completely.

Section 2 has the purpose of presenting the formalism. For the sake of completeness we will also sketch the derivations of the ADK equations. Section 3 is dedicated to an extensive study of second-order Lagrangian equations. In the case of a scalar field one can practically 'solve' the ADK equations establishing a polynomial structure in the second-order derivatives. This central result greatly simplifies the study of (Noetherian) symmetries.

In section 4 we impose, in addition, invariance with respect to some symmetry group. Combining this with the result from section 3 one can completely analyse some interesting

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symmetry groups such as the Poincaré invariance and the so-called universal invariance [6]. Section 5 is dedicated to some final comments.

2. A higher-order Lagrangian formalism

2.1.

The kinematical structure of classical field theory is based on a fibred bundle structure $\pi : S \mapsto M$ where S and M are differentiable manifolds of dimensions $\dim(M) = n, \dim(S) = N + n$ and π is the canonical projection of the fibration. Usually M is interpreted as the 'spacetime' manifold and the fibres of S as the field variables. Next, one considers the k-jet bundle $J^k_n(S) \mapsto M$ $(k = 0, \ldots, p)$. By convention $J^0_n(S) \equiv S$ and $p \in \mathbb{N} \cup \{\infty\}$.

One must usually take $p \in \mathbb{N}$ but sufficiently large such that all formulae make sense. Let us consider a local system of coordinates in the chart $U \subseteq S : (x^\mu)$ $(\mu = 1, \ldots, n)$.

Then on some chart $V \subseteq \pi^{-1}(U) \subset S$ we take a local coordinate system adapted to the fibration structure: (x^μ, ψ^A) $(\mu = 1, \ldots, n, A = 1, \ldots, N)$ such that the canonical projection is $\pi(x^\mu, \psi^A) = (x^\mu)$.

Then one can extend this system of coordinates to $J^k_n(S)$ for any $k \leqslant p$:

$$(x^\mu, \psi^A, \psi^A_\mu, \ldots, \psi^A_{\mu_1,\ldots,mu_k}) \qquad 1 \leqslant \mu_1 \leqslant \cdots \mu_k \leqslant n.$$

If $\mu_1, \ldots, \mu_k$ are arbitrary, then by $\{\mu_1, \ldots, \mu_k\}$ we understand the operation of increasing ordering; then the notation $\psi^A_{\{\mu_1,\ldots,\mu_k\}}$ obviously makes sense.

2.2.

Let us consider $s < p$ and T a $(n+1)$-form which can be written in the local coordinates introduced above as

$$T = \mathcal{T}_A \, \mathrm{d}\psi^A \wedge \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \tag{2.1}$$

with $\mathcal{T}_A$ some smooth functions of $(x^\mu, \psi^A, \psi^A_\mu, \ldots, \psi^A_{\mu_1,\ldots,\mu_s})$.

Then T can be globally defined. Indeed, if we make a change of charts adapted to the fibre bundle structure:

$$\phi(x^\mu, \psi^A) = (f^\mu(x), F^A(x, \psi)) \tag{2.2}$$

then in the new coordinates T has the same structure (2.1) as above. In fact, one immediately obtains

$$\mathcal{T}'_A = \det\left(\frac{\partial f^\mu}{\partial x^\nu}\right) \frac{\partial F^B}{\partial \psi^A} \mathcal{T}_B \circ \dot{\phi} \tag{2.3}$$

where $\dot{\phi}$ is the lift of ϕ to $J^k_n(S)$.

We call such a T a *differential equation of order s*.

2.3.

To introduce some special type of differential equations we need some very useful notation [4]. We define the differential operators:

$$\partial^{\mu_1,\ldots,\mu_l}_A \equiv \frac{r_1! \ldots r_l!}{l!} \frac{\partial}{\partial \psi^A_{\{\mu_1,\ldots,\mu_l\}}} \tag{2.4}$$

for any $l = 0, \ldots, k$. Here r_i is the number of times the index i appears in the sequence $\mu_1, \ldots, \mu_l$. The combinatorial factor in (2.4) avoids possible overcounting in the computations which will appear in the following. One then has

$$\partial_A^{\mu_1,\ldots,\mu_l} \psi^B_{\nu_1,\ldots,\nu_l} = \frac{1}{l!} \delta^A_B \mathrm{perm}\left(\delta^{\mu_i}_{\nu_j}\right) \qquad (\forall l \geqslant 0)$$

and

$$\partial_A^{\mu_1,\ldots,\mu_l} \psi^B_{\nu_1,\ldots,\nu_m} = 0 \qquad (l \neq m)$$

where by perm(A) we mean the permanent of the matrix A.

Next, we define the total derivative operators:

$$D_\mu = \frac{\partial}{\partial x^\mu} + \sum_{l \geqslant 0} \psi^A_{\nu_1,\ldots,\nu_l \mu} \partial_A^{\nu_1,\ldots,\nu_l}. \tag{2.5}$$

One can check that

$$D_\mu \psi^A_{\nu_1,\ldots,\nu_l} = \psi^A_{\nu_1,\ldots,\nu_l \mu} \tag{2.6}$$

$$[D_\mu, D_\nu] = 0. \tag{2.7}$$

Finally we define the differential operators

$$D_{\mu_1,\ldots,\mu_l} \equiv D_{\mu_1} \ldots D_{\mu_l}. \tag{2.8}$$

Because of (2.7) the order of the factors on the right-hand side is irrelevant.

2.4.

A differential equation T is called *locally variational* (or of the *Euler–Lagrange type*) *iff* there exists a local real function $\mathcal{L}$ such that the functions $\mathcal{T}_A$ from (2.1) are of the form:

$$\mathcal{E}_A(\mathcal{L}) \equiv \sum_{l \geqslant 0} (-1)^l D_{\mu_1,\ldots,\mu_l} (\partial_A^{\mu_1,\ldots,\mu_l} \mathcal{L}). \tag{2.9}$$

One calls $\mathcal{L}$ a *local Lagrangian* and

$$L \equiv \mathcal{L}\, \mathrm{d}x^1 \wedge \cdots \mathrm{d}x^n \tag{2.10}$$

a *local Lagrange form*. Let us note that L can be globally defined if we admit that at the change of charts (2.2) $\mathcal{L}$ changes:

$$\mathcal{L}' = \det\left(\frac{\partial f^\mu}{\partial x^\nu}\right) \mathcal{L} \circ \dot{\phi}. \tag{2.11}$$

If the differential equation T is constructed as above then we denote it by $E(L)$. A local Lagrangian is called a *total divergence* if it is of the form:

$$\mathcal{L} = D_\mu V^\mu. \tag{2.12}$$

One can check that in this case we have

$$E(L) = 0. \tag{2.13}$$

This property follows easily from

$$\left[\partial_A^{\mu_1,\ldots,\mu_l}, D_\nu\right] = \frac{1}{l} \sum_{i=1}^{l} \delta^{\mu_i}_\nu \partial_A^{\mu_1,\ldots,\hat{\mu}_i,\ldots,\mu_l} \qquad (\forall l \geqslant 0). \tag{2.14}$$

The converse of this statement is true if one works on $J^\infty_n(S)$ (see [7]). It is not known whether this is true on $J^p_n(S)$ with p finite. A local Lagrangian verifying (2.13) is called *trivial*.

2.5.

Now we come to the central result from [4, 5].

Theorem 1. Let T be a differential equation of order s. Then T is locally variational *iff* the functions $\mathcal{T}_A$ from (2.1) verify the following equations:

$$\partial_A^{\mu_1,\dots,\mu_l}\mathcal{T}_B = \sum_{p=l}^{s}(-1)^p C_p^l D_{\mu_{l+1},\dots,\mu_p}\partial_B^{\mu_1,\dots,\mu_p}\mathcal{T}_A \qquad (l = 0,\dots,s). \quad (2.15)$$

Remark 1. These are the so-called *Anderson–Duchamp–Krupka equations.* For $n = 1$ and $s = 2$ one obtains the well known Helmholtz equations.

Sketch of the proof. [4] We remind the reader that we are working on $J_n^p(S)$ with p sufficiently large.

$\Longrightarrow$:

Suppose that $\mathcal{L}$ is a local Lagrangian depending on $(x^\mu, \psi^A, \psi^A_\mu, \dots, \psi^A_{\mu_1,\dots,\mu_r})$ with $2r \geqslant s$.

Then we must show that $\mathcal{T}_A = \mathcal{E}_A(L)$ verify the ADK equations. The idea is as follows. Let y^A $(A = 1,\dots,N)$ be some x-dependent functions and

$$y^A_{\mu_1,\dots,\mu_l} \equiv \frac{\partial^l y^A}{\partial x^{\mu_1}\dots\partial x^{\mu_l}} \qquad (\forall l = 0,\dots,2r).$$

We define (locally) the vector field Y by

$$Y = \sum_{l=0}^{2r} y^A_{\mu_1,\dots,\mu_l}\partial_A^{\mu_1,\dots,\mu_l}.$$

One proves by direct computations that

$$\mathcal{L}_Y(L) = i_Y E(L) + L_0.$$

Here $\mathcal{L}_Y$ and i_Y are the standard operations of Lie derivative and inner contraction. L_0 is a Lagrange form corresponding to the trivial Lagrangian $\mathcal{L}_0 = D_\mu V^\mu$ where

$$V^\mu \equiv \sum_{p=1}^{r}\sum_{l=1}^{p}(-1)^{l+1} y^A_{\mu_{l+1},\dots,\mu_p} D_{\mu_1,\dots,\mu_{l-1}}\left(\partial_A^{\mu_1,\dots,\hat{\mu}_l,\dots,\mu_p\nu}\mathcal{L}\right).$$

So one has evidently

$$E(\mathcal{L}_Y(L)) = E(i_Y E(L)).$$

But the Euler–Lagrange operator E contains only the operators $\partial_A^{\mu_1,\dots,\mu_l}$ and D_μ (see (2.9)) and one can check directly that both commute with $\mathcal{L}_Y$ when applied to L; so E commutes with $\mathcal{L}_Y$ and the preceding relation implies

$$\mathcal{L}_Y(T) = E(i_Y T).$$

The ADK equations are nothing but the coefficients of $y^A_{\mu_1,\dots,\mu_l}$ in this equation.

$\Longleftarrow$:

Suppose that the differential equation T verifies (locally) the equations (2.15). One can choose the system of local coordinates such that $\mathcal{T}_A$ are regular functions in the point: $\psi^A_{\mu_1,\dots,\mu_l} = 0$ $(l = 0,\dots,s)$. Then one defines the (local) Lagrangian:

$$\mathcal{L} = \int_0^1 \psi^A \mathcal{T}_A \circ \chi_\lambda \, \mathrm{d}\lambda \qquad (2.16)$$

where

$$\chi_\lambda(x^\mu, \psi^A, \psi^A_\mu, \dots, \psi^A_{\mu_1,\dots,\mu_p}) = (x^\mu, \lambda\psi^A, \lambda\psi^A_\mu, \dots, \lambda\psi^A_{\mu_1,\dots,\mu_p}).$$

Then by direct computations one gets that $\mathcal{T}_A = \mathcal{E}_A(L)$. □

Expression (2.16) is called the *Tonti Lagrangian.*

2.6.

One would like to show that the ADK equations have a global meaning, i.e. if in some chart $\mathcal{T}_A$ verifies (2.15), then $\mathcal{T}'_A$ given by (2.3) verifies (2.15). Suppose that $\mathcal{T}_A$ verifies (2.15). Then theorem 1 shows that $\mathcal{T}_A = \mathcal{E}_A(L)$ for some Lagrange form L. If we consider a change of coordinates ϕ on S (see section 2.2) one can prove that

$$\mathcal{T}'_A = \mathcal{E}(L') \tag{2.17}$$

where L' is the Lagrange form associated with the Lagrangian given by (2.11). We apply again theorem 1 and obtain that $\mathcal{T}'_A$ verifies again (2.15). Let us give an idea of the proof of (2.17). An *evolution* is any *section* $\Psi : M \to S$ *of the bundle* $\pi : S \mapsto M$.

Let us denote by $\dot{\Psi} : M \to J^s_n(S)$ the natural lift of Ψ and define the action functional by

$$\mathcal{A}_L(\Psi) \equiv \int (\dot{\Psi})^* L. \tag{2.18}$$

The fundamental formula of the variational calculus is then

$$\delta_X \mathcal{A}_L(\Psi) \equiv \int (\dot{\Psi})^* i_X E(L) \tag{2.19}$$

where $X \equiv X^A \frac{\partial}{\partial \psi^A}$ is the infinitesimal variation. One computes this variation in two obvious ways and discovers that $\mathcal{E}_A(L)$ and $\mathcal{E}_A(L')$ are connected by a relation of the type (2.3). From this the equation (2.17) follows immediately.

2.7.

Let us suppose that T is a differential equation and $\pi : S \mapsto M$ is a evolution. One says that Ψ is a *solution* of T if one has

$$(\dot{\Psi})^* T = 0. \tag{2.20}$$

If T is locally variational $T = E(L)$ one obtains the global form of the *Euler–Lagrange equations*. In local coordinates one can arrange that Ψ has the form $x^\mu \mapsto (x^\mu, \Psi(x))$; then $\dot{\Psi} : M \to J^s_n(S)$ is given by

$$x^\mu \mapsto \left(x^\mu, \Psi(x), \frac{\partial \Psi}{\partial x^\mu}(x), \dots, \frac{\partial \Psi}{\partial x^{\mu_1} \dots \partial x^{\mu_s}}(x)\right)$$

and (2.20) takes the well known form.

2.8.

We now come to the notion of symmetry. By a *symmetry* of T we understand a map $\phi \in \mathrm{Diff}(S)$ such that if $\Psi : M \to S$ is a solution of T, then $\phi \circ \Psi$ is a solution of T also.

It is tempting to try to classify all possible symmetries associated with a given T. In general, this problem is too difficult to tackle. We will solve a particular case in the next section. For the moment we content ourselves with noting that if ϕ verifies

$$(\dot{\phi})^* T = \lambda T \qquad (\lambda \in \mathbb{R}^*) \tag{2.21}$$

($\dot{\phi} \in \mathrm{Diff}(J^s_n(S))$ being the natural lift of ϕ), then ϕ is a symmetry. For $\lambda = 1$ these are the so-called *Noetherian symmetries.* Indeed if $T = E(L)$ one can recover the usual definition:

$$\mathcal{A}_L(\phi \circ \Psi) = \mathcal{A}_L(\Psi) + \text{a trivial action} \tag{2.22}$$

where by a *trivial action* we mean an action $\mathcal{A}_{L_0}$ with L_0 a trivial Lagrangian. Noetherian symmetries are important because from (2.22) one can obtain conservation laws.

3. Second-order Euler–Lagrange equations

3.1.

We particularize the ADK equations for case $s = 2$ of second-order Euler–Lagrange equations. It is not hard to obtain the following set of equations:

$$\partial^{\mu_1\mu_2}_A \mathcal{T}_B = \partial^{\mu_1\mu_2}_B \mathcal{T}_A \tag{3.1}$$

$$\left(\partial^{\mu\rho_1}_B \partial^{\rho_2\rho_3}_C + \partial^{\mu\rho_2}_B \partial^{\rho_3\rho_1}_C + \partial^{\mu\rho_3}_B \partial^{\rho_1\rho_2}_C\right) \mathcal{T}_A = 0 \tag{3.2}$$

$$\partial^{\mu_1}_A \mathcal{T}_B + \partial^{\mu_1}_B \mathcal{T}_A = 2\left(\partial_{\mu_2} + \psi^C_{\mu_2}\partial_C + \psi^C_{\{\mu_2\nu_1\}}\partial^{\nu_1}_C\right)\partial^{\mu_1\mu_2}_B \mathcal{T}_A \tag{3.3}$$

$$\begin{aligned}\partial_A \mathcal{T}_B - \partial_B \mathcal{T}_A = &-\left(\partial_{\mu_1} + \psi^C_{\mu_1}\partial_C + \psi^C_{\{\mu_1\nu_1\}}\partial^{\nu_1}_C\right)\partial^{\mu_1}_B \mathcal{T}_A + \left(\partial_{\mu_1} + \psi^C_{\mu_1}\partial_C + \psi^C_{\{\mu_1\nu_1\}}\partial^{\nu_1}_C\right)\\ &\times \left(\partial_{\mu_2} + \psi^D_{\mu_2}\partial_D + \psi^D_{\{\mu_2\nu_2\}}\partial^{\nu_2}_D\right)\partial^{\mu_1\mu_2}_B \mathcal{T}_A.\end{aligned} \tag{3.4}$$

It is plausible to conjecture that from (3.1) and (3.2) it follows that $\mathcal{T}_A$ is a polynomial in the second-order derivatives $\psi^A_{\{\mu\nu\}}$. We have succeeded in proving this conjecture for the case of the scalar field $N = 1$.

3.2.

Let $M \simeq \mathbb{R}^n$ with coordinates (x^μ) $\mu = 1, \ldots, n$ and $S \subset M \times \mathbb{R}$ with coordinates (x^μ, ψ). We can naturally imbed $J^k_n(S)$ in a Euclidean space with coordinates $(x^\mu, \psi, \psi_\mu, \ldots, \psi_{\mu_1,\ldots,\mu_k})$.

Then we have, from (2.1), (2.4) and (2.5),

$$T = \mathcal{T}\, \mathrm{d}\psi \wedge \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \tag{3.5}$$

$$\partial^{\mu_1,\ldots,\mu_l} \equiv \frac{r_1!\ldots r_l!}{l!}\frac{\partial}{\partial\psi_{\{\mu_1,\ldots,\mu_l\}}} \tag{3.6}$$

$$D_\mu = \frac{\partial}{\partial x^\mu} + \sum_{l\geqslant 0} \psi_{\{\nu_1,\ldots,\nu_l\mu\}}\partial^{\nu_1,\ldots,\nu_l}. \tag{3.7}$$

The ADK equations (3.1)–(3.4) simplify considerably. In fact (3.1) is trivial, and (3.2) and (3.3) become

$$(\partial^{\mu\rho_1}\partial^{\rho_2\rho_3} + \partial^{\mu\rho_2}\partial^{\rho_3\rho_1} + \partial^{\mu\rho_3}\partial^{\rho_1\rho_2})\mathcal{T} \tag{3.8}$$

and

$$\partial^\mu \mathcal{T} = \left(\frac{\partial}{\partial x^\nu} + \psi_\nu \partial + \psi_{\{\nu\rho\}}\partial^\rho\right)\partial^{\mu\nu}\mathcal{T} \tag{3.9}$$

respectively.

Finally (3.4) is a consequence of (3.9). We will be able to prove that (3.8) is compatible with a polynomial structure of T in the second-order derivatives $\psi_{\{\mu\nu\}}$.

3.3.

Let us note that Tonti Lagrangian associated with T is (see (2.16))

$$\mathcal{L} = \int_0^1 \psi \mathcal{T} \circ \chi_\lambda \, d\lambda \tag{3.10}$$

with

$$\chi_\lambda(x^\mu, \psi, \psi_\mu, \psi_{\{\mu\nu\}}) = (x^\mu, \lambda\psi, \lambda\psi_\mu, \lambda\psi_{\{\mu\ nu\}}).$$

The Euler–Lagrange equations for $\mathcal{L}$ are *a priori* of fourth order because the Lagrangian is of second order (see (2.9)). It follows that there are some constraints on $\mathcal{L}$; namely one should require that the third- and fourth-order terms in the expression $E(L)$ should be identically zero. It is easy to prove that this condition amounts to

$$(\partial^{\mu\rho_1}\partial^{\rho_2\rho_3} + \partial^{\mu\rho_2}\partial^{\rho_3\rho_1} + \partial^{\mu\rho_3}\partial^{\rho_1\rho_2})\,\mathcal{L} = 0. \tag{3.11}$$

More precisely we have

Lemma 1. A second-order Lagrangian $\mathcal{L}$ leads to second-order Euler–Lagrange equations if and only if it verifies the relation (3.11). In this case we have

$$\begin{aligned}\mathcal{E}(L) = \partial\mathcal{L} - \left(\frac{\partial}{\partial x^\mu} + \psi_\mu\partial + \psi_{\{\mu\nu\}}\partial^\nu\right)\partial^\mu\mathcal{L} + \left(\frac{\partial}{\partial x^{\mu_1}} + \psi_{\mu_1}\partial + \psi_{\{\mu_1\nu_1\}}\partial^{\nu_1}\right) \\ \times \left(\frac{\partial}{\partial x^{\mu_2}} + \psi_{\mu_2}\partial + \psi_{\{\mu_2\nu_2\}}\partial^{\nu_2}\right)\partial^{\mu_1\mu_2}\mathcal{L}.\end{aligned} \tag{3.12}$$

3.4.

We turn now to the study of the equations (3.8) (or (3.11)). Let us define the expressions:

$$\psi^{\mu_1,\dots,\mu_k;\nu_1,\dots,\nu_k} \equiv \frac{1}{(n-k)!}\varepsilon^{\mu_1,\dots,\mu_n}\varepsilon^{\nu_1,\dots,\nu_n}\prod_{i=k+1}^{n}\psi_{\{\mu_i\nu_i\}} \qquad (\forall k = 0,\dots,n). \tag{3.13}$$

Up to a sign, $\psi^{\mu_1,\dots,\mu_k;\nu_1,\dots,\nu_k}$ is the determinant of the matrix $\psi_{\{\mu\nu\}}$ with the lines $\mu_1,\dots,\mu_k$ and the columns $\nu_1,\dots,\nu_k$ deleted. The combinatorial factor is chosen such that

$$\psi^{\emptyset,\emptyset} = \det(\psi_{\{\mu\nu\}}). \tag{3.14}$$

We prove now

Theorem 2. The general solution of the equations (3.8) is of the following form:

$$\mathcal{T} = \sum_{k=0}^{n}\frac{1}{(k!)^2}\mathcal{T}_{\mu_1,\dots,\mu_k;\nu_1,\dots,\nu_k}\psi^{\mu_1,\dots,\mu_k;\nu_1,\dots,\nu_k} \tag{3.15}$$

where $\mathcal{T}_{\dots}$ are independent of $\psi_{\{\mu\nu\}}$:

$$\partial^{\rho\sigma}\mathcal{T}_{\mu_1,\dots,\mu_k;\nu_1,\dots,\nu_k} = 0 \qquad (\forall k = 0,\dots,n) \tag{3.16}$$

and have the same symmetry properties as $\psi^{\cdots}$: complete antisymmetry in $\mu_1,\dots,\mu_k$, complete antisymmetry in $\nu_1,\dots,\nu_k$ and symmetry with respect to the interchange: $\mu_1,\dots,\mu_k \leftrightarrow \nu_1,\dots,\nu_k$.

Proof.

(i) One uses induction over n. For $n = 2$, equations (3.8) are simple to write and one indeed obtains that the general solution is of the form (3.15). We suppose that we have the assertion of the theorem for a given n and we prove it for $n+1$. In this case the indices $\mu, \nu, \ldots$ take values (for notational convenience) $\mu, \nu, \ldots = 0, \ldots, n$ and $i, j, \ldots = 1, \ldots, n$. If we consider in (3.8) that $\mu, \rho_1, \rho_2, \rho_3 = 1, \ldots, n$ then we can apply the induction hypothesis and we get

$$\mathcal{T} = \sum_{k=0}^{n} \frac{1}{(k!)^2} \tilde{\mathcal{T}}_{i_1,\ldots,i_k;j_1,\ldots,j_k} \tilde{\psi}^{i_1,\ldots,i_k;j_1,\ldots,j_k}. \tag{3.17}$$

Here $\tilde{\mathcal{T}}$ has obvious symmetry properties and can depend on x, ψ, ψ_μ *and* $\psi_{\{0\mu\}}$. The minors $\tilde{\psi}^{\cdots}$ are constructed from the matrix $\psi_{\{ij\}}$ according to the prescription (3.13).

(ii) We still have at our disposal the relation (3.8) where at least one index takes the value zero. We obtain rather easily that

$$\left(\partial^{00}\right)^2 \tilde{\mathcal{T}}_{i_1,\ldots,i_k;j_1,\ldots,j_k} = 0 \qquad (\forall k = 0, \ldots, n) \tag{3.18}$$

$$\partial^{00}\partial^{0l}\tilde{\mathcal{T}}_{i_1,\ldots,i_k;j_1,\ldots,j_k} = 0 \qquad (\forall k = 0, \ldots, n) \tag{3.19}$$

$$\partial^{0l}\partial^{0m}\tilde{\mathcal{T}}_{\emptyset;\emptyset} = 0 \tag{3.20}$$

$$\frac{1}{2}\sum_{p,q=1}^{k}(-1)^{p+q}\left(\delta^m_{i_p}\delta^l_{j_q} + \delta^l_{i_p}\delta^m_{j_q}\right)\partial^{00}\tilde{\mathcal{T}}_{i_1,\ldots,\hat{i}_p,\ldots i_k;j_1,\ldots,\hat{j}_q,\ldots,j_k} + 2\partial^{0l}\partial^{0m}\tilde{\mathcal{T}}_{i_1,\ldots,i_k;j_1,\ldots,j_k} = 0$$
$$(\forall k = 1, \ldots, n) \tag{3.21}$$

$$\sum_{(l,m,r)}\sum_{p,q=1}^{k}(-1)^{p+q}\left(\delta^m_{i_p}\delta^r_{j_q} + \delta^r_{i_p}\delta^m_{j_q}\right)\partial^{0l}\tilde{\mathcal{T}}_{i_1,\ldots,\hat{i}_p,\ldots i_k;j_1,\ldots,\hat{j}_q,\ldots,j_k} = 0 \qquad (\forall k = 1, \ldots, n). \tag{3.22}$$

Here by $\sum_{(l,m,r)}$ we understand the sum over all cyclic permutations of the indices l, m, r.

It is a remarkable fact that these equations can be solved, i.e. one can describe the most general solution.

From (3.18) we have

$$\tilde{\mathcal{T}}_{i_1,\ldots,i_k;j_1,\ldots,j_k} = \mathcal{T}^{(0)}_{i_1,\ldots,i_k;j_1,\ldots,j_k} + \psi_{\{00\}}\mathcal{T}^{(1)}_{i_1,\ldots,i_k;j_1,\ldots,j_k} \qquad (k = 0, \ldots, n) \tag{3.23}$$

with the restrictions

$$\partial^{00}\mathcal{T}^{(l)}_{i_1,\ldots,i_k;j_1,\ldots,j_k} = 0 \qquad (k = 0, \ldots, n;\ l = 0, 1). \tag{3.24}$$

From (3.19) and (3.20) we also get

$$\partial^{0l}\mathcal{T}^{(1)}_{i_1,\ldots,i_k;j_1,\ldots,j_k} = 0 \qquad (k = 0, \ldots, n) \tag{3.25}$$

$$\partial^{0l}\partial^{0m}\mathcal{T}^{(0)}_{\emptyset;\emptyset} = 0. \tag{3.26}$$

Finally (3.21) and (3.22) become

$$\frac{1}{2}\sum_{p,q=1}^{k}(-1)^{p+q}\left(\delta^m_{i_p}\delta^l_{j_q} + \delta^l_{i_p}\delta^m_{j_q}\right)\partial^{00}\mathcal{T}^{(1)}_{i_1,\ldots,\hat{i}_p,\ldots i_k;j_1,\ldots,\hat{j}_q,\ldots,j_k} + 2\partial^{0l}\partial^{0m}\mathcal{T}^{(0)}_{i_1,\ldots,i_k;j_1,\ldots,j_k} = 0$$
$$(\forall k = 1, \ldots, n) \tag{3.27}$$

$$\sum_{(l,m,r)}\sum_{p,q=1}^{k}(-1)^{p+q}\left(\delta^m_{i_p}\delta^r_{j_q} + \delta^r_{i_p}\delta^m_{j_q}\right)\partial^{0l}\mathcal{T}^{(0)}_{i_1,\ldots,\hat{i}_p,\ldots i_k;j_1,\ldots,\hat{j}_q,\ldots,j_k} = 0 \qquad (\forall k = 1, \ldots, n). \tag{3.28}$$

(iii) Now the analysis can be pushed further if we apply the operator ∂^{0r} to (3.27); taking into account (3.24) we obtain

$$\partial^{0r}\partial^{0l}\partial^{0m}\mathcal{T}^{(0)}_{i_1,\ldots,i_k;j_1,\ldots,j_k} = 0 \qquad (\forall k = 1,\ldots,n). \tag{3.29}$$

Equations (3.26) and (3.29) can be used to obtain easily a polynomial structure in $\psi_{\{0l\}}$. The details are elementary and one gets, from (3.26),

$$\mathcal{T}^{(0)}_{\emptyset;\emptyset} = \mathcal{T}^{\emptyset}_{\emptyset;\emptyset} + \sum_l \mathcal{T}^{l}_{\emptyset;\emptyset}\psi_{\{0l\}} \tag{3.30}$$

with

$$\partial^{0\mu}\mathcal{T}^{\cdots}_{\emptyset;\emptyset} = 0. \tag{3.31}$$

Analogously, one establishes from (3.29) that

$$\mathcal{T}^{(0)}_{i_1,\ldots,i_k;j_1,\ldots,j_k} = \mathcal{T}^{\emptyset}_{i_1,\ldots,i_k;j_1,\ldots,j_k} + \sum_l \mathcal{T}^{l}_{i_1,\ldots,i_k;j_1,\ldots,j_k}\psi_{\{0l\}} + \frac{1}{2}\sum_{l,m}\mathcal{T}^{lm}_{i_1,\ldots,i_k;j_1,\ldots,j_k}\psi_{\{0l\}}\psi_{\{0m\}}$$
$$(\forall k = 1,\ldots,n). \tag{3.32}$$

Here we have

$$\partial^{0\mu}\mathcal{T}^{\cdots}_{i_1,\ldots,i_k;j_1,\ldots,j_k} = 0 \qquad (k = 0,\ldots,n). \tag{3.33}$$

We can also suppose that

$$\mathcal{T}^{lm}_{i_1,\ldots,i_k;j_1,\ldots,j_k} = \mathcal{T}^{ml}_{i_1,\ldots,i_k;j_1,\ldots,j_k}. \tag{3.34}$$

If we insert (3.32) into (3.27) we get

$$\mathcal{T}^{lm}_{i_1,\ldots,i_k;j_1,\ldots,j_k} = -\frac{1}{2}\sum_{p,q=1}^{k}(-1)^{p+q}\left(\delta^m_{i_p}\delta^l_{j_q} + \delta^l_{i_p}\delta^m_{j_q}\right)\mathcal{T}^{(1)}_{i_1,\ldots,\hat{i}_p,\ldots i_k;j_1,\ldots,\hat{j}_q,\ldots,j_k}. \tag{3.35}$$

Finally, inserting (3.32) into (3.28) we get

$$\sum_{(l,m,r)}\sum_{p,q=1}^{k}(-1)^{p+q}\left(\delta^m_{i_p}\delta^r_{j_q} + \delta^r_{i_p}\delta^m_{j_q}\right)\mathcal{T}^{l}_{i_1,\ldots,\hat{i}_p,\ldots i_k;j_1,\ldots,\hat{j}_q,\ldots,j_k} = 0 \qquad (\forall k = 1,\ldots,n) \tag{3.36}$$

$$\sum_{(l,m,r)}\sum_{p,q=1}^{k}(-1)^{p+q}\left(\delta^m_{i_p}\delta^r_{j_q} + \delta^r_{i_p}\delta^m_{j_q}\right)\mathcal{T}^{ls}_{i_1,\ldots,\hat{i}_p,\ldots i_k;j_1,\ldots,\hat{j}_q,\ldots,j_k} = 0 \qquad (\forall k = 1,\ldots,n). \tag{3.37}$$

Let us summarize what we have obtained up until now. The solution of (3.18)–(3.22) is given by (3.23) where $\mathcal{T}^{(0)}_{\ldots}$ is given by (3.32) with $\mathcal{T}^{lm}_{\ldots}$ explicitated by (3.35) and $\mathcal{T}^{l}_{\ldots}$ restricted by (3.36). One also has to keep in mind (3.31) and (3.33). We will show that (3.35) identically verifies (3.37) so in fact we are left to solve only (3.36).

(iv) It is rather strange that equations of the type (3.36) and (3.37) can be analysed using techniques characteristic to quantum mechanics, namely the machinery of the Fock space. In fact, let us consider the antisymmetric Fock space $\mathcal{F}^{(-)}(\mathbb{R}^n)$; we define next the Hilbert space $\mathcal{H} \equiv \mathcal{F}^{(-)}(\mathbb{R}^n) \otimes \mathcal{F}^{(-)}(\mathbb{R}^n)$.

It is clear that the tensors $\mathcal{T}^{\cdots}_{i_1,\ldots,i_k;j_1,\ldots,j_k}$ can be viewed as elements of $\mathcal{H}$ also verifying the symmetry property

$$S\mathcal{T}^{\cdots} = \mathcal{T}^{\cdots} \tag{3.38}$$

where

$$S(\phi \otimes \psi) = \psi \otimes \phi \qquad (\forall \phi, \psi \in \mathcal{F}^{(-)}(\mathbb{R}^n)). \tag{3.39}$$

Let us denote by $a_{(l)}$ and $a^{*(l)}$ ($\forall l = 1, \ldots, n$) the annihilation and the creation operators acting in $\mathcal{F}^{(-)}(\mathbb{R}^n)$; then we have in $\mathcal{H}$ the operators

$$b^{*(l)} \equiv a^{*(l)} \otimes \mathbf{1} \qquad c^{*(l)} \equiv \mathbf{1} \otimes a^{*(l)}$$

and similarly for $b_{(l)}$ and $c_{(l)}$. In this notation (3.35)–(3.37) become

$$\mathcal{T}^{lm} = -\frac{1}{2k^2}\left[b^{*(l)}c^{*(m)} + b^{*(m)}c^{*(l)}\right]\mathcal{T}^{(1)} \tag{3.40}$$

$$\sum_{(l,m,r)}\left[b^{*(l)}c^{*(m)} + b^{*(m)}c^{*(l)}\right]\mathcal{T}^{r} = 0 \tag{3.41}$$

$$\sum_{(l,m,r)}\left[b^{*(l)}c^{*(m)} + b^{*(m)}c^{*(l)}\right]\mathcal{T}^{rs} = 0. \tag{3.42}$$

Now it is extremely easy to prove that (3.40) identically verifies (3.42) so, in fact, (3.35) identically verifies (3.37) as we have announced above.

We concentrate now on (3.41). If we take $l = m = r$ we obtain

$$b^{*(l)}c^{*(l)}\mathcal{T}^{l} = 0 \qquad \text{(no summation over } l!)$$

which easily implies that $\mathcal{T}^l$ must have the following structure:

$$\mathcal{T}^l = b^{*(l)}B + c^{*(l)}C + b^{*(l)}c^{*(l)}D$$

with B, C and D obtained from the vacuum by applying polynomial operators in all creation operators with the exception of $b^{*(l)}$ and $c^{*(l)}$.

From (3.38) we obtain

$$C = SB \qquad D = SD$$

so, in fact, $\mathcal{T}^l$ is of the form

$$\mathcal{T}^l = b^{*(l)}B' + c^{*(l)}SB' \tag{3.43}$$

with B' arbitrary. Now it is easy to prove that (3.43) identically verifies (3.41) so it is the most general solution of this equation.

Reverting to index notation, it follows that the most general solution of (3.36) is of the form:

$$\mathcal{T}^l_{i_1,\ldots,i_k;j_1,\ldots,j_k} = \sum_{p=1}^{k}(-1)^{p-1}\left(\delta^l_{i_p}\mathcal{T}_{i_1,\ldots,\hat{i_p},\ldots i_k;j_1,\ldots,j_k} + \delta^l_{j_p}\mathcal{T}_{j_1,\ldots,\hat{j_p},\ldots,j_k;i_1,\ldots,i_k}\right) \tag{3.44}$$

where $\mathcal{T}_{i_2,\ldots,i_k;j_1,\ldots,j_k}$ is completely antisymmetric in $i_2, \ldots, i_k$ and in $j_1, \ldots, j_k$.

The structure of $\mathcal{T}_{i_1,\ldots,i_k;j_1,\ldots,j_k}$ is completely elucidated: it is given by (3.23) where $\mathcal{T}^{(0)}_{\ldots}$ is given by (3.30) and (3.32); in (3.32) $\mathcal{T}^l_{\ldots}$ is given by (3.44) and $\mathcal{T}^{lm}_{\ldots}$ by (3.35). Everything depends on some arbitrary functions $\mathcal{T}^{\ldots}_{\emptyset;\emptyset}$ and $\mathcal{T}^{(1)}_{\ldots}$ which do not depend on $\psi_{\{0\mu\}}$.

(v) It remains to introduce the expression for $\tilde{\mathcal{T}}$ in (3.17) and regroup the terms. If we define:

$$\mathcal{T}_{\emptyset;\emptyset} \equiv \mathcal{T}^{\emptyset}_{\emptyset;\emptyset}$$

$$\mathcal{T}_{0,i_1,\ldots,i_k;0j_1,\ldots,j_k} \equiv (k+1)\mathcal{T}^{\emptyset}_{i_1,\ldots,i_k;j_1,\ldots,j_k}$$

$$\mathcal{T}_{i_1,\ldots,i_k;j_1,\ldots,j_k} \equiv \mathcal{T}^{(1)}_{i_1,\ldots,i_k;j_1,\ldots,j_k}$$

$$\mathcal{T}_{0,i_2,\ldots,i_k;j_1,\ldots,j_k} \equiv k\mathcal{T}_{i_2,\ldots,i_k;j_1,\ldots,j_k}$$

then (3.17) goes into (3.15). $\square$

3.5.

We can insert the solution (3.15) of (3.8) into (3.9) and obtain some restrictions on the functions $\mathcal{T}_{\dots}$.

It is convenient to define

$$\frac{\delta}{\delta x^\mu} \equiv \frac{\partial}{\partial x^\mu} + \psi_\mu \partial. \tag{3.45}$$

Then we obtain

$$\sum_{i,j=1}^{k}(-1)^{i+j}\left(\delta^\rho_{\mu_i}\frac{\delta\mathcal{T}_{\mu_1,\dots,\hat{\mu}_i,\dots,\mu_k;\nu_1,\dots,\hat{\nu}_j,\dots,\nu_k}}{\delta x^{\nu_j}} + \delta^\rho_{\nu_j}\frac{\delta\mathcal{T}_{\mu_1,\dots,\hat{\mu}_i,\dots,\mu_k;\nu_1,\dots,\hat{\nu}_j,\dots,\nu_k}}{\delta x^{\mu_i}}\right)$$
$$= \sum_{i=1}^{k}(-1)^{i-1}\left[\delta^\rho_{\mu_i}\left(\partial^\lambda\mathcal{T}_{\lambda,\mu_1,\dots,\hat{\mu}_i,\dots,\mu_k;\nu_1,\dots,\nu_k}\right) + \delta^\rho_{\nu_i}\left(\partial^\lambda\mathcal{T}_{\mu_1,\dots,\mu_k;\lambda,\nu_1,\dots,\hat{\nu}_i,\dots,\nu_k}\right)\right]$$
$$(k = 1,\dots,n-1) \tag{3.46}$$

$$\partial^\rho\mathcal{T}_{\mu_1,\dots,\mu_n;\nu_1,\dots,\nu_n} = \frac{1}{2}\sum_{i,j=1}^{n}(-1)^{i+j}\left(\delta^\rho_{\mu_i}\frac{\delta\mathcal{T}_{\mu_1,\dots,\hat{\mu}_i,\dots,\mu_n;\nu_1,\dots,\hat{\nu}_j,\dots,\nu_n}}{\delta x^{\nu_j}}\right.$$
$$\left.+\delta^\rho_{\nu_j}\frac{\delta\mathcal{T}_{\mu_1,\dots,\hat{\mu}_i,\dots,\mu_n;\nu_1,\dots,\hat{\nu}_i,\dots,\nu_n}}{\delta x^{\mu_i}}\right). \tag{3.47}$$

So we have

Theorem 3. The most general local variational second-order differential equation for a scalar field is given by (3.15) where the functions $\mathcal{T}_{\dots}$ have the structure decribed in the statement of theorem 2 and also verify (3.46) and (3.47).

3.6.

We concentrate now on the form of possible Lagrangians producing second-order differential equations. According to section 3.3 such a Lagrangian can be taken to be of second order and constrained by (3.11). According to theorem 2 this means that $\mathcal{L}$ can be taken in the form

$$\mathcal{L} = \sum_{k=0}^{n}\frac{1}{(k!)^2}\mathcal{L}_{\mu_1,\dots,\mu_k;\nu_1,\dots,\nu_k}\psi^{\mu_1,\dots,\mu_k;\nu_1,\dots,\nu_k} \tag{3.48}$$

with $\mathcal{L}_{\dots}$ independent of $\psi_{\{\rho\sigma\}}$:

$$\partial^{\rho\sigma}\mathcal{L}_{\mu_1,\dots,\mu_k;\nu_1,\dots,\nu_k} = 0 \qquad (k = 0,\dots,n) \tag{3.49}$$

and with the same symmetry properties as $\mathcal{T}_{\dots}$.

Of course, it is possible that two different Lagrangians of the type (3.48) give the same Euler–Lagrange operator. To investigate the extent of this arbitrariness we compute $E(L)$. As expected, we get something of the form (3.15):

$$\mathcal{E}(L) = \sum_{k=0}^{n}\frac{1}{(k!)^2}\mathcal{T}(L)_{\mu_1,\dots,\mu_k;\nu_1,\dots,\nu_k}\psi^{\mu_1,\dots,\mu_k;\nu_1,\dots,\nu_k} \tag{3.50}$$

where

$$\mathcal{T}(L)_{\mu_1,\dots,\mu_k;\nu_1,\dots,\nu_k} = (n-k+1)\partial\mathcal{L}_{\mu_1,\dots,\mu_k;\nu_1,\dots,\nu_k} + \frac{\delta}{\delta x^\rho}\left(\partial^\rho\mathcal{L}_{\mu_1,\dots,\mu_k;\nu_1,\dots,\nu_k}\right)$$
$$+\sum_{i=1}^{k}(-1)^i\left[\frac{\delta}{\delta x^{\mu_i}}\left(\partial^\lambda\mathcal{L}_{\lambda,\mu_1,\dots,\hat{\mu}_i,\dots,\mu_k;\nu_1,\dots,\nu_k}\right) + \frac{\delta}{\delta x^{\nu_i}}\left(\partial^\lambda\mathcal{L}_{\mu_1,\dots,\mu_k;\lambda\nu_1,\dots,\hat{\nu}_i,\dots,\nu_k}\right)\right]$$

$$+\sum_{i,j=1}^{k}(-1)^{i+j}\frac{\delta^2\mathcal{L}_{\mu_1,\dots,\hat{\mu}_i,\dots,\mu_k;\nu_1,\dots,\hat{\nu}_j,\dots,\nu_k}}{\delta x^{\mu_i}\delta x^{\nu_j}}-\partial^\lambda\partial^\zeta\mathcal{L}_{\lambda,\mu_1,\dots,\mu_k;\zeta,\nu_1,\dots,\nu_k}$$

$$(k=0,\dots,n-2) \tag{3.51}$$

$$\begin{aligned}\mathcal{T}(L)_{\mu_1,\dots,\mu_{n-1};\nu_1,\dots,\nu_{n-1}} &= 2\partial\mathcal{L}_{\mu_1,\dots,\mu_{n-1};\nu_1,\dots,\nu_{n-1}}+\frac{\delta}{\delta x^\rho}\left(\partial^\rho\mathcal{L}_{\mu_1,\dots,\mu_{n-1};\nu_1,\dots,\nu_{n-1}}\right)\\ &+\sum_{i=1}^{n-1}(-1)^i\left[\frac{\delta}{\delta x^{\mu_i}}\left(\partial^\lambda\mathcal{L}_{\lambda,\mu_1,\dots,\hat{\mu}_i,\dots,\mu_{n-1};\nu_1,\dots,\nu_{n-1}}\right)\right.\\ &\left.+\frac{\delta}{\delta x^{\nu_i}}\left(\partial^\lambda\mathcal{L}_{\mu_1,\dots,\mu_{n-1};\lambda,\nu_1,\dots,\hat{\nu}_i,\dots,\nu_{n-1}}\right)\right]\\ &+\sum_{i,j=1}^{n-1}(-1)^{i+j}\frac{\delta^2\mathcal{L}_{\mu_1,\dots,\hat{\mu}_i,\dots,\mu_{n-1};\nu_1,\dots,\hat{\nu}_j,\dots,\nu_{n-1}}}{\delta x^{\mu_i}\delta x^{\nu_j}}\end{aligned} \tag{3.52}$$

$$\begin{aligned}\mathcal{T}(L)_{\mu_1,\dots,\mu_n;\nu_1,\dots,\nu_n} &= \partial\mathcal{L}_{\mu_1,\dots,\mu_n;\nu_1,\dots,\nu_n}-\frac{\delta}{\delta x^\rho}\left(\partial^\rho\mathcal{L}_{\mu_1,\dots,\mu_n;\nu_1,\dots,\nu_n}\right)\\ &+\sum_{i,j=1}^{n}(-1)^{i+j}\frac{\delta^2\mathcal{L}_{\mu_1,\dots,\hat{\mu}_i,\dots,\mu_n;\nu_1,\dots,\hat{\nu}_j,\dots,\nu_n}}{\delta x^{\mu_i}\delta x^{\nu_j}}.\end{aligned} \tag{3.53}$$

We use in these equations the Bourbaki convention $\sum_\emptyset\cdots=0$. So, $\mathcal{L}$ given by (3.48) leads to trivial Euler–Lagrange equations *iff* the expressions $\mathcal{T}(L)_{\dots}$ defined above are identically zero.

3.7.

We are now prepared to investigate the most general expression of a symmetry for a second-order local variational differential equation for a scalar field. We have:

Theorem 4. Let T a local variational differential equation for a scalar field and $\phi\in\mathrm{Diff}(S)$ a symmetry. Then there exists $\rho\in\mathcal{F}(J^s_n(S))$ such that

$$\partial^{\mu\nu}\rho=0 \tag{3.54}$$

and

$$(\dot{\phi})^*\,T=\rho T. \tag{3.55}$$

Proof. The condition that ϕ is a symmetry is that (2.20) should be equivalent to the same equation with $\Psi\mapsto\phi\circ\Psi$ for any evolution $\Psi: M\mapsto S$. Because Ψ is arbitrary one obtains that

$$T=0\Leftrightarrow(\dot{\phi})^*T=0.$$

Equivalently, if we define $\mathcal{T}'$ by

$$(\dot{\phi})^*T=\mathcal{T}'\,\mathrm{d}\psi\wedge\mathrm{d}x^1\wedge\ldots\wedge\mathrm{d}x^n$$

then we have

$$\mathcal{T}=0\Leftrightarrow\mathcal{T}'=0.$$

One easily obtains from here, under some reasonable regularity conditions, that there exists a function $\rho\in\mathcal{F}(J^s_n(S))$ such that

$$\mathcal{T}'=\rho\,\mathcal{T}. \tag{3.56}$$

Because T and T' are locally variational $\mathcal{T}$ and $\mathcal{T}'$ have the polynomial structure given by (3.15). So we have $\rho = p/p'$, where p and p' are some polynomials in $\psi_{\{\mu\nu\}}$. So, (3.56) is

$$p\mathcal{T} = p'\mathcal{T}'. \tag{3.57}$$

We identify the terms of maximal degree in $\psi_{\{\mu\nu\}}$ in both sides and find

$$p_{\max}\mathcal{T}_{\emptyset;\emptyset}\det(\psi) = p'_{\max}\mathcal{T}'_{\emptyset;\emptyset}\det(\psi) \Leftrightarrow p_{\max} = \rho_0\ p'_{\max} \tag{3.58}$$

where $\rho_0 \equiv \mathcal{T}_{\emptyset;\emptyset}/\mathcal{T}'_{\emptyset;\emptyset}$. We insert this in (3.57) and continue by recurrence. Finally one gets $\rho = \rho_0$ so we have in fact (3.54). Moreover, it is clear that (3.56) is equivalent to (3.55).□

Remark 2. One can obtain some useful relations from (3.56) if we insert it into (3.46) and (3.47) and take into account the fact that $\mathcal{T}$ verify these equations also. One obtains

$$\begin{aligned}
&\sum_{i,j=1}^{k}(-1)^{i+j}\left(\delta^{\rho}_{\mu_i}\frac{\delta f}{\delta x^{\nu_j}} + \delta^{\rho}_{\nu_j}\frac{\delta f}{\delta x^{\mu_i}}\right)\mathcal{T}_{\mu_1,\ldots,\hat{\mu}_i,\ldots,\mu_k;\nu_1,\ldots,\hat{\nu}_j,\ldots,\nu_k}\\
&\quad = \sum_{i=1}^{k}(-1)^{i-1}\left(\delta^{\rho}_{\mu_i}\mathcal{T}_{\lambda,\mu_1,\ldots,\hat{\mu}_i,\ldots,\mu_k;\nu_1,\ldots,\nu_k} + \delta^{\rho}_{\nu_i}\mathcal{T}_{\mu_1,\ldots,\mu_k;\lambda,\nu_1,\ldots,\hat{\nu}_i,\ldots,\nu_k}\right)\frac{\partial f}{\partial\psi_\lambda}\\
&\quad (k = 1,\ldots,n-1)
\end{aligned} \tag{3.59}$$

$$\frac{\partial f}{\partial\psi^{\rho}}\mathcal{T}_{\mu_1,\ldots,\mu_n;\nu_1,\ldots,\nu_n} = \frac{1}{2}\sum_{i,j=1}^{n}(-1)^{i+j}\left(\delta^{\rho}_{\mu_i}\frac{\delta f}{\delta x^{\nu_j}} + \delta^{\rho}_{\nu_j}\frac{\delta f}{\delta x^{\mu_i}}\right)\mathcal{T}_{\mu_1,\ldots,\hat{\mu}_i,\ldots,\mu_n;\nu_1,\ldots,\hat{\nu}_j,\ldots,\nu_n}. \tag{3.60}$$

These relations can be used to obtain some restrictions on the function f. For instance, let us suppose that $\frac{\delta f}{\delta x^{\lambda}} = 0$ and $\frac{\partial f}{\partial\psi^{\rho}} = 0$. Then one obtains that either $\frac{\partial f}{\partial\psi} = 0$ (in this case f is locally constant) or $\mathcal{T}_{\ldots}$ verifies

$$\sum_{i,j=1}^{k}(-1)^{i+j}\left(\delta^{\rho}_{\mu_i}\psi_{\nu_j} + \delta^{\rho}_{\nu_j}\psi_{\mu_i}\right)\mathcal{T}_{\mu_1,\ldots,\hat{\mu}_i,\ldots,\mu_k;\nu_1,\ldots,\hat{\nu}_j,\ldots,\nu_k} = 0 \qquad (k = 1,\ldots,n). \tag{3.61}$$

Remark 3. Theorem 2 is a sort of Lee-Hwa Chung theorem [9] for the Lagrangian formalism.

4. Lagrangian systems with groups of symmetries

4.1.

We will study two types of symmetry in this section. First, the case when the group of symmetries is a Lie group (with a typical case the Poincaré invariance) and next the case when the group of symmetries is infinite dimensional (with the typical case the universal invariance).

4.2.

Let us consider a second-order locally variational equation with Poincaré invariance. (When speaking of Poincaré invariance we will have in mind the proper orthochronous Poincaré group, although there is no difficulty in treating the inversions with the same method.)

So, M from section 3.2 is the n-dimensional Minkowski space and for obvious reasons the indices $\mu, \nu, \ldots$ will take the values $0, 1, \ldots, n-1$; the Minkowski bilinear form $G_{..}$ has the signature $(1, -1, \ldots, -1)$. The action of the Poincaré group on $S \equiv M \times \mathbb{R}$ is

$$\phi_{L,a}(x,\psi) = (Lx + a, \psi) \tag{4.1}$$

with L a Lorentz transformation and $a \in \mathbb{R}^n$ a translation in the affine space M. The lift of (4.1) to $J^2_n(S)$ is

$$\dot\phi_{L,a}(x, \psi, \psi_\mu, \psi_{\{\mu\nu\}}) = (Lx + a, \psi, L_\mu{}^\nu \psi_\nu, L_\mu{}^\rho L_\nu{}^\sigma \psi_{\{\rho\sigma\}}) \tag{4.2}$$

and the condition of Poincaré invariance is, by definition,

$$(\dot\phi_{L,a})^* \, T = T \tag{4.3}$$

(so we are considering only Noetherian symmetries).

The equation (4.3) is equivalent to

$$\mathcal{T} \circ \dot\phi_{L,a} = \mathcal{T}. \tag{4.4}$$

For $L = \mathbf{1}$ one obtains the x-independence of $\mathcal{T}$:

$$\frac{\partial \mathcal{T}}{\partial x^\mu} = 0 \tag{4.5}$$

and from (4.4) we still have the Lorentz invariance of $\mathcal{T}$:

$$\mathcal{T}(\psi, L_\mu{}^\rho L_\nu{}^\sigma \psi_{\{\rho\sigma\}}) = \mathcal{T}(\psi, \psi_\mu, \psi_{\{\mu\nu\}}). \tag{4.6}$$

If we insert (3.15) into (4.5) and (4.6) we get that the $\mathcal{T}_{\dots}$ are x-independent:

$$\frac{\partial \mathcal{T}_{\mu_1,\dots,\mu_k;\nu_1,\dots,\nu_k}}{\partial x^\lambda} = 0 \qquad (k = 0, \dots, n) \tag{4.7}$$

and also that $\mathcal{T}_{\dots}$ are Lorentz covariant tensors depending only on ψ and ψ_μ.

Using the usual method [8] of analysing the generic form of such a tensorial covariant functions one obtains that $\mathcal{T}_{\dots}$ is a sum of expressions of the type

$$\psi_. \dots \psi_. G_{..} \dots G_{..} \mathcal{A}(\psi, J)$$

where $J \equiv \psi^\mu \psi_\mu$ is a Lorentz invariant.

One now has to take into account the various symmetry properties of $\mathcal{T}_{\dots}$. First one notices that one cannot have more than two factors $\psi_.$ because for three factors or more one contradicts the antisymmetry in $\mu_1, \dots, \mu_k$ or/and in $\nu_1, \dots, \nu_k$. Because we also have symmetry with respect to the change $(\mu_1, \dots, \mu_k) \leftrightarrow (\nu_1, \dots, \nu_k)$ it is clear that we have two types of term: terms containing no $\psi_.$ factors and terms containing exactly two $\psi_.$ factors, more precisely of the form $\psi_{\mu_k}\psi_{\nu_l}$. Also, to avoid contradiction of the antisymmetry the allowed factors $G_{..}$ are of the form $G_{\mu_k\nu_l}$.

Summing up, the most general Lorentz covariant tensor $\mathcal{T}_{\dots}$ respecting the symmetry properties from the statement of theorem 2 is

$$\mathcal{T}_{\mu_1,\dots,\mu_k;\nu_1,\dots,\nu_k} = \mathcal{A}_k I_{\mu_1,\dots,\mu_k;\nu_1,\dots,\nu_k} + \mathcal{B}_k J_{\mu_1,\dots,\mu_k;\nu_1,\dots,\nu_k}. \tag{4.8}$$

Here $\mathcal{A}_k$ and $\mathcal{B}_k$ are smooth functions of ψ and J. We use the convention $\mathcal{B}_0 = 0$ and we have defined

$$I_{\mu_1,\dots,\mu_k;\nu_1,\dots,\nu_k} = \sum_{\sigma,\tau \in \mathcal{P}_{\{1,\dots,k\}}} (-1)^{|\sigma|+|\tau|} \prod_{i=1}^k G_{\mu_{\sigma(i)}\nu_{\tau(i)}} \qquad (k = 0, \dots, n) \tag{4.9}$$

$$J_{\mu_1,\dots,\mu_k;\nu_1,\dots,\nu_k} = \sum_{\sigma,\tau \in \mathcal{P}_{\{1,\dots,k\}}} (-1)^{|\sigma|+|\tau|} \psi_{\mu_{\sigma(1)}} \psi_{\nu_{\tau(1)}} \prod_{i=2}^k G_{\mu_{\sigma(i)}\nu_{\tau(i)}} \qquad (k = 0, \dots, n) \tag{4.10}$$

with the conventions $I_{\emptyset;\emptyset} \equiv 1$, $J_{\emptyset;\emptyset} \equiv 0$.

One must insert (4.8) into the remaining ADK equations (3.46) and (3.47). The result of this tedious computation is

$$\frac{\partial \mathcal{A}_{k-1}}{\partial \psi} - 2k\frac{\partial \mathcal{A}_k}{\partial J} - 2J\frac{\partial \mathcal{B}_k}{\partial J} - (n+k)\mathcal{B}_k = 0 \qquad (k = 1, \ldots, n-1) \tag{4.11}$$

$$\frac{\partial \mathcal{B}_k}{\partial \psi} = 0 \ (k = 1, \ldots, n-2) \tag{4.12}$$

$$2\frac{\partial \mathcal{A}_n}{\partial J} + (n-1)!\frac{\partial \mathcal{A}_{n-1}}{\partial \psi} = 0 \tag{4.13}$$

where we understand that for $n = 2$, (4.12) disappears. Inserting (4.8) into (3.15) it follows that we have:

Theorem 5. The most general local variational second-order differential equation for a scalar field having Poincaré invariance in the sense (4.4) is of the form

$$\mathcal{T} = \mathcal{A}_0 \det(\psi) + \sum_{k=1}^{n-1} (\mathcal{A}_k I_k + \mathcal{B}_k J_k) + \mathcal{A}_n \tag{4.14}$$

where I_k and J_k are the Lorentz invariants:

$$I_k \equiv \left(\prod_{i=1}^{k} G_{\mu_i \nu_i} \right) \psi^{\mu_1,\ldots,\mu_k;\nu_1,\ldots,\nu_k} \tag{4.15}$$

and

$$J_k \equiv \psi_{\mu_1}\psi_{\nu_1} \left(\prod_{i=2}^{k} G_{\mu_i \nu_i} \right) \psi^{\mu_1,\ldots,\mu_k;\nu_1,\ldots,\nu_k}. \tag{4.16}$$

Also the functions $\mathcal{A}_0, \ldots, \mathcal{A}_n$ and $\mathcal{B}_1, \ldots, \mathcal{B}_{n-1}$ depend smoothly only on ψ and J and verify the equations (4.11)–(4.13). One can take $\mathcal{B}_1, \ldots, \mathcal{B}_{n-2}$ arbitrary functions of J and $\mathcal{A}_n, \mathcal{B}_{n-1}$ arbitrary functions of ψ and J. Then (4.11)–(4.13) can be used to fix $\mathcal{A}_0, \ldots, \mathcal{A}_{n-1}$ up to an arbitrary function of J. The Tonti Lagrangian has the structure (4.14) also.

4.3.

Let us now study the so-called universal invariance. Suppose $F \in \mathrm{Diff}(\mathbb{R})$; then we define $\phi_F \in \mathrm{Diff}(S)$ by

$$\phi_F(x, \psi) = (x, F(\psi)). \tag{4.17}$$

The natural lift of $\phi_F \in \mathrm{Diff}(S)$ to $J_n^2(S)$ is

$$\dot{\phi}_F(x, \psi, \psi_\mu, \psi_{\{\mu\nu\}}) = (x, F(\psi), F'(\psi)\psi_\mu, F'(\psi)\psi_{\{\mu\nu\}} + F''(\psi)\psi_\mu\psi_\nu). \tag{4.18}$$

We say that the differential equation T has *universal invariance* if we have

$$(\dot{\phi}_F)^* T = \rho_F T. \tag{4.19}$$

The function $\rho_F \in \mathrm{Diff}(J_n^2(S))$ does not depend on $\psi_{\{\mu\nu\}}$ according to theorem 4 and it is a cohomological object [10]. As in [10] we will consider only the case when

$$\rho_F = (F')^p. \tag{4.20}$$

In this case (4.19) is equivalent to

$$\mathcal{T} \circ \dot{\phi}_F = (F')^{p-1}\mathcal{T}. \tag{4.21}$$

Remark 4. According to remark 2, we have two cases: either $p = 1$ or we have (3.61).

We take F to be an infinitesimal diffeomorphism, i.e.

$$F(\psi) = \psi + \theta(\psi) \tag{4.22}$$

with θ infinitesimal but otherwise arbitrary and we can cast (4.21) into the infinitesimal form; one obtains

$$\partial \mathcal{T} = 0 \tag{4.23}$$

$$\psi_\mu \partial^\mu \mathcal{T} + \psi_\mu \psi_\nu \partial^{\mu\nu} \mathcal{T} = (p-1)\mathcal{T} \tag{4.24}$$

$$\psi_\mu \psi_\nu \partial^{\mu\nu} \mathcal{T} = 0. \tag{4.25}$$

Let us note that (4.24) is the infinitesimal form of the homogeneity equation:

$$\mathcal{T}(x, \lambda\psi_\mu, \lambda\psi_{\{\mu\nu\}}) = \lambda^{p-1}\mathcal{T}(x, \psi_\mu, \psi_{\{\mu\nu\}}) \qquad (\forall \lambda \in \mathbb{R}^*). \tag{4.26}$$

If we insert in these equations the expression (3.15) we obtain, equivalently,

$$\partial \mathcal{T}_{\mu_1,\ldots,\mu_k;\nu_1,\ldots,\nu_k} = 0 \qquad (k = 0, \ldots, n) \tag{4.27}$$

$$\mathcal{T}_{\mu_1,\ldots,\mu_k;\nu_1,\ldots,\nu_k}(x, \lambda\psi_\mu) = \lambda^{k+p-1-n}\mathcal{T}_{\mu_1,\ldots,\mu_k;\nu_1,\ldots,\nu_k}(x, \psi_\mu) \qquad (k = 0, \ldots, n) \tag{4.28}$$

$$\sum_{i,j=1}^{n} (-1)^{i+j} \psi_{\mu_i} \psi_{\nu_j} \mathcal{T}_{\mu_1,\ldots,\hat{\mu}_i,\ldots,\mu_k;\nu_1,\ldots,\hat{\nu}_j,\ldots,\nu_k} = 0 \qquad (k = 1, \ldots, n). \tag{4.29}$$

Let us note that for $p \neq 0$, (4.29) follows from (3.61).

One must add to these equations (3.46) and (3.47) which, in our case, are

$$\begin{aligned}
\sum_{i,j=1}^{k} (-1)^{i+j} &\left(\delta^\rho_{\mu_i} \frac{\partial \mathcal{T}_{\mu_1,\ldots,\hat{\mu}_i,\ldots,\mu_k;\nu_1,\ldots,\hat{\nu}_j,\ldots,\nu_k}}{\partial x^{\nu_j}} + \delta^\rho_{\nu_j} \frac{\partial \mathcal{T}_{\mu_1,\ldots,\hat{\mu}_i,\ldots,\mu_k;\nu_1,\ldots,\hat{\nu}_j,\ldots,\nu_k}}{\partial x^{\mu_i}} \right) \\
&= \sum_{i=1}^{k} (-1)^{i-1} \Big[\delta^\rho_{\mu_i} \left(\partial^\lambda \mathcal{T}_{\lambda,\mu_1,\ldots,\hat{\mu}_i,\ldots,\mu_k;\nu_1,\ldots,\nu_k} \right) \\
&\quad + \delta^\rho_{\nu_i} \left(\partial^\lambda \mathcal{T}_{\mu_1,\ldots,\mu_k;\lambda,\nu_1,\ldots,\hat{\nu}_i,\ldots,\ nu_k} \right) \Big] \qquad (k = 1, \ldots, n-1)
\end{aligned} \tag{4.30}$$

$$\begin{aligned}
\partial^\rho \mathcal{T}_{\mu_1,\ldots,\mu_n;\nu_1,\ldots,\nu_n} = \frac{1}{2} \sum_{i,j=1}^{n} (-1)^{i+j} &\left(\delta^\rho_{\mu_i} \frac{\partial \mathcal{T}_{\mu_1,\ldots,\hat{\mu}_i,\ldots,\mu_n;\nu_1,\ldots,\hat{\nu}_j,\ldots,\nu_n}}{\partial x^{\nu_j}} \right. \\
&\left. + \delta^\rho_{\nu_j} \frac{\partial \mathcal{T}_{\mu_1,\ldots,\hat{\mu}_i,\ldots,\mu_n;\nu_1,\ldots,\hat{\nu}_j,\ldots,\nu_n}}{\partial x^{\mu_i}} \right).
\end{aligned} \tag{4.31}$$

The system (4.27)–(4.31) seems to be too hard to solve in the general case. We content ourselves with studying two particular cases.

(i) T is translational invariant, i.e.

$$\frac{\partial \mathcal{T}}{\partial x^\lambda} = 0 \tag{4.32}$$

or

$$\frac{\partial \mathcal{T}_{\mu_1,\ldots,\mu_k;\nu_1,\ldots,\nu_k}}{\partial x^\lambda} = 0 \qquad (k = 0, \ldots, n). \tag{4.33}$$

For $p = n + 1$ one obtains the particular solution

$$\mathcal{T} = \mathcal{T}_{\emptyset;\emptyset} \det(\psi) \tag{4.34}$$

with $\mathcal{T}_{\emptyset;\emptyset}$ constant. This is the solution appearing in [6].

(ii) It is clear that T follows from a first-order Lagrangian *iff*

$$\mathcal{T}_{\mu_1,\ldots,\mu_k;\nu_1,\ldots,\nu_k} = 0 \qquad (k = 0, \ldots, n-2). \tag{4.35}$$

In this case:

$$\mathcal{T} = \mathcal{T}_0 + \mathcal{T}^{\rho\sigma}\psi_{\{\rho\sigma\}}. \tag{4.36}$$

One easily obtains that (4.27)–(4.31) reduces to

$$\partial\mathcal{T}_0 = 0 \tag{4.37}$$

$$\partial\mathcal{T}^{\rho\sigma} = 0 \tag{4.38}$$

$$\mathcal{T}_0(x, \lambda\psi_\mu) = \lambda^{p-1}\mathcal{T}_0(x, \psi_\mu) \tag{4.39}$$

$$\mathcal{T}^{\rho\sigma}(x, \lambda\psi_\mu) = \lambda^{p-2}\mathcal{T}^{\rho\sigma}(x, \psi_\mu) \tag{4.40}$$

$$\psi_\rho\psi_\sigma\mathcal{T}^{\rho\sigma} = 0 \tag{4.41}$$

$$\partial^\rho\mathcal{T}^{\mu\nu} - \partial^\mu\mathcal{T}^{\rho\nu} = 0 \tag{4.42}$$

$$\partial^\rho\mathcal{T}_0 = \frac{\partial\mathcal{T}^{\rho\sigma}}{\partial x^\sigma}. \tag{4.43}$$

This system was analysed in [10] where it was found that it has solutions for $p = 0$ and $p = 1$.

5. Conclusions

The central formula obtained in this paper is (3.15). This expression affords a complete treatement of local variational second-order differential equations with groups of symmetry.

It is plausible that (3.15) admits generalizations for the case $N > 1$ (i.e. fields with more than one component) and for $s > 2$ (i.e. equations of arbitrary order). Maybe as a first step one should try the more modest cases: $N > 1, s = 2$ or $N = 1, s > 2$.

These problems will be addressed in future publications.

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